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NOTE

A Note on General Algorithm for Variational Inequalities

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In this note we introduce and study a new general strongly quasivariational inequality problem which includes, as special cases, a number of known classes of variational inequalities studied previously by many authors in this field. A general and unified iterative algorithm for finding the approximate solution to this problem is constructed by the projection method. We prove the existence of the solution for this problem and the convergence of the iterative sequence generated by this algorithm.

1. INTRODUCTION

Variational inequality theory is a very powerful tool of current mathematical technology and has become an interesting and fascinating branch of applicable mathematics. This theory has been used in the study of a wide class of problems arising in contact problems in elasticity, fluid flow through porous media, and general equilibrium of transportation and economics in a general and unified framework. Many authors have presented a substantial number of numerical methods for the numerical solutions of variational inequalities; see [1] for more details. As one of the most effective numerical techniques, the idea of Glowinski, Lions, and Tremolieres [1] has been modified and extended by Noor [2, 3] to prove the existence of solutions for variational inequality problems. Cohen [4] has extended this technique. Recently, to solve the variational inequality problem (2.1) in [5], Noor introduced and studied a new general auxiliary variational inequality problem, and proposed and analyzed a quite general

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algorithm for the problem (2.1) in [5], using the technique of Cohen [4]. But he mentioned that the projection method has not been being extended for the problem (2.1) in [5] due to the presence of the form $b(u, v)$.

In this note, motivated and inspired by the research work in [5], we introduce and study a new general strongly quasivariational inequality problem which includes, as special cases, a number of known classes of variational inequalities studied previously by many authors in this field. A general and unified iterative algorithm for finding the approximate solutions to this problem is constructed by the projection method. We prove the existence of the solution for this problem, and the convergence of the iterative sequence generated by this algorithm. Our results are the improvements and extension of the results obtained previously by many authors including Noor [5].

2. FORMULATION AND BASIC RESULTS

Let H be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $K \subset H$ be a nonempty closed convex subset, R be the set of all real numbers, and the form $b(\cdot, \cdot)$: $H \times H \to R$ be nondifferentiable and satisfy the following properties:

(1) $b(u, v)$ is linear in the first argument.

(2) $b(u, v)$ is bounded, that is, there exists a constant $\nu > 0$ such that

$$|b(u, v)| \leq \nu \|u\| \, \|v\| \qquad \text{for all } u, v \in H. \tag{2.1}$$

(3) $b(u, v) - b(u, w) \leq b(u, v - w)$ for all $u, v, w \in H$.

(4) $b(u, v)$ is convex in the second argument.

We need the following concepts.

DEFINITION 2.1. An operator T: $H \to H$ is called

(a) *Strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\| \qquad \text{for all } u, v \in H.$$

(b) *Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\| \qquad \text{for all } u, v \in H.$$

Let A, T, and g be nonlinear mappings from H into itself. Then we consider a problem of finding $u \in K$ such that $g(u) \in K$ and

$$\begin{aligned} \langle A(g(u)), v - g(u) \rangle &+ \rho b(u, v) - \rho b(u, g(u)) \\ &\geq \langle A(u), v - g(u) \rangle - \rho \langle Tu, v - g(u) \rangle \end{aligned} \tag{2.2}$$

for all $v \in K$ and a constant $\rho > 0$, which is called the general strongly variational inequality problem.

If we denote $g(u)$ by w, then problem (2.2) is similar to the general auxiliary variational inequality problem considered and studied by Noor [5], which is to find a unique $w \in K$ for some $u \in K$ such that

$$\begin{aligned}\langle A(w), v-w\rangle + \rho b(u,v) - \rho b(u,w) \\ \geq \langle A(u), v-w\rangle - \rho\langle Tu, v-w\rangle \end{aligned} \tag{$*$}$$

for all $v \in K$ and a constant $\rho > 0$.

Next we give an important and useful special case of problem (2.2) [i.e., Problem 2.1 in [5], which Noor [5] was devoted to consider and study by introducing the above auxiliary problem $(*)$]:

Find $u \in K$ such that

$$\langle Tu, v-u\rangle + b(u,v) - b(u,u) \geq 0 \qquad \text{for all } v \in K \tag{2.3}$$

where the form $b(\cdot,\cdot)$: $H \times H \to R$ is nondifferentiable and satisfies the above properties (1)–(4).

Remark 2.1. For appropriate and suitable choice of the operators T, A, and g, a number of known classes of variational inequalities can be obtained as special cases of problem (2.2) studied previously by many authors including Lions and Stampacchia [6], Duvaut and Lions [7], Glowinski, Lions, and Tremolieres [1], Noor [3], and Cohen [4].

If the convex set K depends upon the solution, i.e., K is a point-to-set mapping from H into itself, then problem (2.2) becomes the general strongly quasivariational inequality problem of finding $u \in K$ such that $g(u) \in K(u)$ and

$$\begin{aligned}\langle A(g(u)), v-g(u)\rangle + \rho b(u,v) - \rho b(u, g(u)) \\ \geq \langle A(u), v-g(u)\rangle - \rho\langle Tu, v-g(u)\rangle \end{aligned} \tag{2.4}$$

for all $v \in K(u)$ and a constant $\rho > 0$.

In many important applications, the set $K(u)$ is of the form

$$K(u) = m(u) + K, \tag{2.5}$$

where m is a point-to-point mapping and K is a closed convex set [8].

To prove our main result, we need the following lemmas:

LEMMA 2.1 [9]. *Let $K \subset H$ be a closed convex subset. Then given $z \in H$, we have*

$$u = P_K z$$

if and only if $u \in K$ and

$$\langle u-z, v-u\rangle \geq 0 \qquad \textit{for all } v \in K,$$

where P_K is a projection of H onto K.

LEMMA 2.2 [9]. *P_K is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\| \qquad \text{for all } u, v \in H.$$

LEMMA 2.3 [10]. *If $K(u)$ is of type (2.5), then for each $u, v \in H$,*

$$P_{K(u)} v = m(u) + P_K(v - m(u)).$$

LEMMA 2.4. *Let $K(u)$ be of type (2.5). Then $u \in K$ is a solution of the problem (2.4) if and only if $u \in K$ satisfies $g(u) \in K(u)$ and*

$$\langle u - \varphi(u), v - g(u) \rangle \geq 0 \qquad \text{for all } v \in K(u), \tag{2.6}$$

where $\varphi(u)$: $H \to H$ and for some constant $\rho > 0$,

$$\langle \varphi(u), v \rangle = \langle u, v \rangle - \rho\langle Tu, v \rangle - \rho b(u, v) + \langle [A \circ (I - g)](u), v \rangle \tag{2.7}$$

for all $v \in K(u)$.

Remark 2.2. *The mapping $A \circ (I - g)$ is defined as*

$$[A \circ (I - g)](x) = A(x) - A(g(x)) \qquad \text{for each } x \in H.$$

Proof of Lemma 2.4. Let $u \in K$ be a solution of problem (2.4). Then we derive $g(u) \in K(u)$ and

$$\begin{aligned} \langle A(g(u)), v - g(u) \rangle &+ \rho b(u, v) - \rho b(u, g(u)) \\ &\geq \langle A(u), v - g(u) \rangle - \rho\langle Tu, v - g(u) \rangle \end{aligned} \tag{2.8}$$

for all $v \in K(u)$ and a constant $\rho > 0$. Using (2.7) and (2.8), we obtain

$$\begin{aligned} &\langle u - \varphi(u), v - g(u) \rangle \\ &\quad = \langle u, v - g(u) \rangle - \langle \varphi(u), v \rangle + \langle \varphi(u), g(u) \rangle \\ &\quad = \langle u, v \rangle - \langle u, g(u) \rangle \\ &\qquad - [\langle u, v \rangle - \rho\langle Tu, v \rangle - \rho b(u, v) + \langle [A \circ (I - g)](u), v \rangle] \\ &\qquad + \langle u, g(u) \rangle - \rho\langle Tu, g(u) \rangle - \rho b(u, g(u)) \\ &\qquad\qquad + \langle A \circ (I - g)](u), g(u) \rangle \\ &\quad = \langle [A \circ (I - g)](u), g(u) - v \rangle + \rho b(u, v) - \rho b(u, g(u)) \\ &\qquad + \rho\langle Tu, v - g(u) \rangle \geq 0 \qquad \text{for all } v \in K(u), \end{aligned}$$

which infers that (2.6) holds.

Conversely, let $u \in K$ satisfy $g(u) \in K(u)$ and (2.6). Then we have

$$\begin{aligned}
\langle u, v - g(u) \rangle
&\geq \langle \varphi(u), v - g(u) \rangle \\
&= \langle \varphi(u), v \rangle - \langle \varphi(u), g(u) \rangle \\
&= \langle u, v \rangle - \rho \langle Tu, v \rangle - \rho b(u, v) + \langle [A \circ (I - g)](u), v \rangle \\
&\quad - \big[\langle u, g(u) \rangle - \rho \langle Tu, g(u) \rangle - \rho b(u, g(u)) \\
&\qquad\qquad + \langle [A \circ (I - g)](u), g(u) \rangle \big] \\
&= \langle u, v - g(u) \rangle - \rho \big[\langle Tu, v - g(u) \rangle + b(u, v) - b(u, g(u)) \big] \\
&\quad + \langle [A \circ (I - g)](u), v - g(u) \rangle \qquad \text{for all } v \in K(u).
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\langle A(g(u)), v - g(u) \rangle + \rho b(u, v) - \rho b(u, g(u)) \\
&\qquad \geq \langle A(u), v - g(u) \rangle - \rho \langle Tu, v - g(u) \rangle \qquad \text{for all } v \in K(u).
\end{aligned}$$

Thus, $u \in K$ is a solution of problem (2.4).

LEMMA 2.5. *Let $K(u)$ be defined as* (2.5). *Then $u \in K$ is a solution of the problem* (2.4) *if and only if $u \in K$ satisfies $g(u) \in K(u)$ and*

$$g(u) = m(u) + P_K(g(u) - u + \varphi(u) - m(u)), \tag{2.9}$$

where m: $H \to H$ and $\varphi(u)$ is defined as (2.7).

Proof. By Lemma 2.4, $u \in K$ is a solution of the problem (2.4) if and only if (2.6) holds. We deduce

$$\langle g(u) - [g(u) - (u - \varphi(u))], v - g(u) \rangle = \langle u - \varphi(u), v - g(u) \rangle \geq 0$$

for all $v \in K(u)$. Hence, by Lemmas 2.1 and 2.3 (2.6) holds if and only if $u \in K$ satisfies $g(u) \in K(u)$ and

$$\begin{aligned}
g(u) &= P_{K(u)}(g(u) - u + \varphi(u)) \\
&= m(u) + P_K(g(u) - u + \varphi(u) - m(u)).
\end{aligned}$$

LEMMA 2.6. *Let the mapping T: $H \to H$ be strongly monotone with constant α and Lipschitz continuous with Lipschitz constant β, A: $H \to H$ be Lipschitz continuous with Lipschitz constant ξ, g: $H \to H$ be Lipschitz continuous with Lipschitz constant σ, and the form $b(u, v)$ satisfy the properties* (1) *and* (2). *Then for any constant $\rho > 0$, there exists*

$$\theta = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + \rho\nu + \xi(1 + \sigma) > 0$$

such that

$$\|\varphi(u_1) - \varphi(u_2)\| \le \theta\|u_1 - u_2\| \qquad \textit{for all } u_1, u_2 \in H,$$

where $\varphi(u)$ *is defined as* (2.7).

Proof. For each $u_1, u_2 \in H$, by (2.7) and property (1) of $b(u,v)$, we obtain

$$\begin{aligned}\langle \varphi(u_1) - \varphi(u_2), v\rangle \\ &= \langle u_1 - u_2, v\rangle - \rho\langle Tu_1 - Tu_2, v\rangle - \rho b(u_1 - u_2, v) \\ &\quad + \langle [A\circ(I-g)](u_1) - [A\circ(I-g)](u_2), v\rangle.\end{aligned}$$

It then follows from (2.1) and the Lipschitz continuity of the operators A and g that

$$\begin{aligned}|\langle \varphi(u_1) - \varphi(u_2), v\rangle| &= |\langle u_1 - u_2 - \rho(Tu_1 - Tu_2), v\rangle \\ &\quad - \rho b(u_1 - u_2, v) + \langle A(u_1) - A(u_2) \\ &\quad - A(g(u_1)) - A(g(u_2)), v\rangle| \\ &\le \|u_1 - u_2 - \rho(Tu_2 - Tu_2)\|\,\|v\| \\ &\quad + \rho\nu\|u_1 - u_2\|\,\|v\| + (\xi + \xi\sigma)\|u_1 - u_2\|\,\|v\|.\end{aligned}$$

Using the Lipschitz continuity and the strong monotonicity of T, we have

$$\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \le \left(1 - 2\alpha\rho + \beta^2\rho^2\right)\|u_1 - u_2\|^2.$$

Hence, we get

$$\begin{aligned}|\langle \varphi(u_1) - \varphi(u_2), v\rangle| &\le \left[\sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + \rho\nu + \xi(1+\sigma)\right] \\ &\quad \cdot \|u_1 - u_2\|\,\|v\| \\ &= \theta\|u_1 - u_2\|\,\|v\|,\end{aligned}$$

which infers that $\|\varphi(u_1) - \varphi(u_2)\| \le \theta\|u_1 - u_2\|$.

3. MAIN RESULT

In this section, we shall suggest and propose a general and unified iterative algorithm for finding the approximate solutions of the general strongly quasivariational inequality problem (2.4) and prove the approximate solutions converge strongly to the exact solution of the problem (2.4).

ALGORITHM 3.1. Given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K(g(u_n) - u_n + \varphi(u_n) - m(u_n)), \qquad n = 0, 1, 2, \ldots,$$

where $\rho > 0$ is a constant.

THEOREM 3.1. *Let the mapping T: $H \to H$ be strongly monotone with constant α and Lipschitz continuous with Lipschitz constant β, A: $H \to H$ be Lipschitz continuous with Lipschitz constant ξ, g: $H \to H$ be Lipschitz continuous with Lipschitz constant σ and strongly monotone with constant δ, and m: $H \to H$ be Lipschitz continuous with Lipschitz constant η. Let the form $b(u, v)$ satisfy the properties* (1) *and* (2). *Assume further that*

$$k < 1/2 \quad \text{and} \quad \alpha > \nu(1 - 2k) + \sqrt{(\beta^2 - \nu^2)4k(1-k)},$$

where $k = \sqrt{1 - 2\delta + \sigma^2} + \eta + \xi(1 + \sigma)/2$. Then the problem (2.4) *has a unique solution u^*, and for each constant $\rho > 0$ with*

$$\left| \rho - \frac{\alpha + \nu(2k - 1)}{\beta^2 - \nu^2} \right| \leq \frac{\sqrt{[\alpha - \nu(1 - 2k)]^2 - (\beta^2 - \nu^2)4k(1-k)}}{\beta^2 - \nu^2},$$

the iterative sequence $\{u_n\}$ generated by Algorithm 3.1 *converges strongly to u^*.*

Proof. By Lemma 2.5, $u \in H$ is a solution of the problem (2.4) if and only if $u \in H$ satisfies (2.9). The mapping F: $H \to H$ is defined by

$$F(u) = u - g(u) + m(u) + P_K(g(u) - u + \varphi(u) - m(u)) \qquad \text{for all } u \in H,$$

where $\varphi(u)$ is defined as (2.7). For each $u_1, u_2 \in H$, we have

$$\begin{aligned}
\|Fu_1 - Fu_2\| &= \|u_1 - g(u_1) + m(u_1) + P_K(g(u_1) - u_1 + \varphi(u_1) - m(u_1)) \\
&\quad - u_2 + g(u_2) - m(u_2) - P_K(g(u_2) - u_2 + \varphi(u_2) - m(u_2))\| \\
&\leq \|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\quad + \|P_K(g(u_1) - u_1 + \varphi(u_1) - m(u_1)) \\
&\quad - P_K(g(u_2) - u_2 + \varphi(u_2) - m(u_2))\|.
\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}
&\|Fu_1 - Fu_2\| \\
&\quad \le \|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\qquad + \|(g(u_1) - u_1 + \varphi(u_1) - m(u_1)) \\
&\qquad - (g(u_2) - u_2 + \varphi(u_2) - m(u_2))\| \\
&\quad \le 2\|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\qquad + \|\varphi(u_1) - \varphi(u_2)\|. \qquad (3.1)
\end{aligned}$$

Using the method of Noor [8], the Lipschitz continuity of g and m, and the strong monotonicity of g, we obtain

$$\begin{aligned}
&\|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\quad \le \|u_1 - u_2 - (g(u_1) - g(u_2))\| + \|m(u_1) - m(u_2)\| \\
&\quad \le \left[\sqrt{1 - 2\delta + \sigma^2} + \eta\right]\|u_1 - u_2\|. \qquad (3.2)
\end{aligned}$$

It follows from (3.1) and (3.2) and Lemma 2.6 that

$$\begin{aligned}
&\|Fu_1 - Fu_2\| \\
&\quad \le 2\left[\sqrt{1 - 2\delta + \sigma^2} + \eta\right]\|u_1 - u_2\| \\
&\qquad + \left[\sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + \rho\nu + \xi(1 + \sigma)\right]\|u_1 - u_2\| \\
&\quad = 2\left[\sqrt{1 - 2\delta + \sigma^2} + \eta + \xi(1 + \sigma)/2\right]\|u_1 - u_2\| \\
&\qquad + \left[\sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + \rho\nu\right]\|u_1 - u_2\| \\
&\quad = (2k + \theta_1)\|u_1 - u_2\|,
\end{aligned}$$

where $\theta_1 = \sqrt{1 + 2\alpha\rho + \beta^2\rho^2} + \rho\nu$ and

$$k = \sqrt{1 - 2\delta + \sigma^2} + \eta + \xi(1 + \sigma)/2.$$

Since $\alpha \le \beta$, $k < 1/2$, and $\alpha > \nu(1 - 2k) + \sqrt{(\beta^2 - \nu^2)4k(1 - k)}$, this implies that for each $\rho > 0$ with

$$\left|\rho - \frac{\alpha + \nu(2k - 1)}{\beta^2 - \nu^2}\right| \le \frac{\sqrt{[\alpha - \nu(1 - 2k)]^2 - (\beta^2 - \nu^2)4k(1 - k)}}{\beta^2 - \nu^2},$$

we have $2k + \theta_1 = 2k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + \rho\nu < 1$. So, F is a contraction mapping. We imply that F has a unique fixed point $u^* \in H$, that is,

$$u^* = u^* - g(u^*) + m(u^*) + P_K(g(u^*) - u^* + \varphi(u^*) - m(u^*)).$$

By Lemma 2.5, u^* is a unique solution of the problem (2.4). Since $u^* \in H$ satisfies (2.9), i.e.,

$$g(u^*) = m(u^*) + P_K(g(u^*) - u^* + \varphi(u^*) - m(u^*)), \quad (3.3)$$

by (3.3) and Algorithm 3.1, we obtain

$$\begin{aligned}
&\|u_{n+1} - u^*\| \\
&\quad \le \|u_n - u^* - (g(u_n) - g(u^*)) + m(u_n) - m(u^*)\| \\
&\qquad + \|P_K(g(u_n) - u_n + \varphi(u_n) - m(u_n)) \\
&\qquad - P_K(g(u^*) - u^* + \varphi(u^*) - m(u^*))\| \\
&\quad \le 2\|u_n - u^* - (g(u_n) - g(u^*)) + m(u_n) - m(u^*)\| \\
&\qquad + \|\varphi(u_n) - \varphi(u^*)\| \\
&\quad \le (2k + \theta_1)\|u_n - u^*\| \\
&\quad \le (2k + \theta_1)^n\|u_1 - u^*\|.
\end{aligned}$$

Noting that $2k + \theta_1 < 1$, we know that $\{u_n\}$ converges strongly to u^*.

Remark 3.1. Theorem 3.1 is the improvement and extension of the results obtained previously by many authors including Lions and Stampacchia [6], Duvaut and Lions [7], Glowinski, Lions, and Tremolieres [1], Cohen [4], and Noor [3, 5]. Especially, our Theorem 3.1 extends the existence of the solution in Theorem 3.1 in [5] to the case of the general strongly quasivariational inequality problem (2.4). Also, Algorithm 3.1 is a very general and unified algorithm for finding the approximate solutions of Problem 2.1 in [5]. This indicates that the projection method is still an effective and useful method for solving variational inequality problems and quasivariational inequality problems.

REFERENCES

1. R. Glowinski, J. L. Lions, and R. Tremolieres, "Numerical Analysis of Variational Inequalities," North-Holland, Amsterdam, 1981.
2. M. A. Noor, General nonlinear variational inequalities, *J. Math. Anal. Appl.* **126** (1987), 78–84.

3. M. A. Noor, Variational inequalities related with a Signorini problem, *C. R. Math. Rep. Acad. Sci. Canada* **7** (1985), 267–272.
4. G. Cohen, Auxiliary problem principle extended to variational inequalities, *J. Optim. Theory Appl.* **59** (1988), 325–333.
5. M. A. Noor, General algorithm for variational inequalities (I), *Math. Japon.* **38** (1993), 47–53.
6. J. L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.
7. G. Duvaut and J. L. Lions, ‘‘Les Inequations en Mechanique et en Physique,’’ Dunod, Paris, 1972.
8. M. A. Noor, Quasivariational inequalities, *Appl. Math. Lett.* **1** (1988), 367–370.
9. D. Kinderlehrar and G. Stampacchia, ‘‘An Introduction to Variational Inequalities and Their Applications,’’ Academic Press, New York, 1980.
10. M. A. Noor, An iterative scheme for a class of quasi variational inequalities, *J. Math. Anal. Appl.* **110** (1985), 463–468.